

Cherry flows with non trivial attractors

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Abstract

We provide an example of Cherry flow (*i.e.* \mathcal{C}^∞ flow on the 2-dimensional torus with a sink and a saddle) having quasi-minimal set which is an attractor. The first return map for such a flow, constructed also in the paper, is a \mathcal{C}^∞ circle map having a flat interval and a non-trivial wandering interval.

1 Introduction

1.1 Discussion and statement of results

In 1986 Poincaré conjectured the existence of a \mathcal{C}^∞ flow on the 2-dimensional torus with a non-trivial minimal set. The conjecture was later disproved by Denjoy. In 1937 Cherry proved that the conjecture of Poincaré is true if you ask the existence of a non-trivial quasi-minimal set. He constructed \mathcal{C}^∞ flows on the torus, called Cherry flows, without closed trajectories and with two singularities, a sink and a saddle, both hyperbolic. The quasi-minimal set in this case is the complement of the stable manifold of the sink. The geometrical properties of the quasi-minimal set have been studied in [6], [10], [12]. In all the known examples the quasi-minimal set has zero Lebesgue measure and almost every orbit converges to the sink. In this paper we construct examples of Cherry flows having quasi-minimal sets which are non-trivial attractors. These examples have two coexisting attractors: the sink and the quasi-minimal set.

The construction is based on the study of the first return map for a Cherry flow and on one of the fundamental question in circle dynamics: whether a circle map is conjugate to a rotation. One of the first steps in this area was done by Denjoy [2]. He proved that any \mathcal{C}^1 diffeomorphism with irrational rotation number and with derivative of bounded variation is conjugate to a rotation. In the same paper he showed that the hypothesis on the derivative is essential by giving examples of \mathcal{C}^1 diffeomorphisms with irrational rotation number which are not conjugate to a rotation. The reason for which the Denjoy function is not conjugate to a rotation is the presence of a wandering interval. A wandering interval I has the property that, for all $n \in \mathbb{Z}$, $f^n(I) \cap I = \emptyset$ and $f|_I^n$ is a diffeomorphism. Since then examples of this kind, called Denjoy counterexamples, have attracted attention of many mathematicians.

Another important contribution given by Katok in [5] is a Denjoy counterexample which is \mathcal{C}^∞ everywhere with the exception of one point which is a non-flat critical point of the function. A \mathcal{C}^∞ circle function being not conjugate to a rotation and with at most two flat critical points was constructed by Hall in [4]. In [9] the author of this paper generalized Hall's construction showing a piece-wise \mathcal{C}^∞ Denjoy counterexample having a flat half-critical point.

Our aim is to further extend the ideas of [4] and [9]. We construct a \mathcal{C}^∞ Denjoy counterexample having an arc of critical points all being flat. Our result is the following:

Theorem 1.1. *For any irrational number $\rho \in [0, 1)$ there exists a \mathcal{C}^∞ , non-decreasing circle map f of degree one and an arc U of the circle such that:*

- $f(U)$ is a point,
- f has rotation number ρ ,
- f is a Denjoy counterexample.

Notice that the arc U is not a wandering interval for f , the forward iterates are not diffeomorphisms. We would like to stress that the techniques developed here and in [9] are quite robust and can be used to produce more counterexamples. We give more details in Appendix.

Our result not only contributes to the field of Denjoy counterexamples but also can be applied to study the quasi-minimal set¹ of Cherry flows.

Using Theorem 1.1 we are now able to study more subtle topological properties of the quasi-minimal set. We recall that any Cherry flow has a well defined rotation number $\rho \in [0, 1)$ equal to the rotation number of its first return map to a chosen Poincaré section and we state:

Theorem 1.2. *For any irrational number $\rho \in [0, 1)$ there exists a Cherry flow with rotation number ρ and with quasi-minimal set which is an attractor. The basin of attraction of the quasi-minimal set has non-empty interior.*

Thanks to this result we are now able to control the long-time behavior of the orbits of a large class of Cherry flows.

An interesting problem arising at this point is understanding physical measures of the flows constructed in Theorem 1.2. Observe that in the standard case of Cherry flows with a saddle point and an attractive point the answer to this question is easy: the Dirac delta at the attractive point is the only physical measure. For the flows of Theorem 1.2 all questions remain open. For example we expect that physical measures are non-trivial measures concentrated on the attractor. For the reference we mention that the only studies of non-trivial physical measures concerned the case of inverted Cherry flows, viz. flows with a saddle point and a repulsive point, see [14], [13], [15].

1.2 Notations and Definitions

By $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ we denote the unit circle and by $R_\rho : \mathbb{S}^1 \mapsto \mathbb{S}^1$, $\rho \in \mathbb{R}$, the map defined as

$$R_\rho(\theta + \mathbb{Z}) = (\theta + \rho + \mathbb{Z})$$

¹The closure of any non-trivial recurrent trajectories is called a quasi-minimal set.

and called rotation by ρ .

Let $\pi : \mathbb{R} \mapsto \mathbb{S}^1$ be the projection of the real line to the circle and let $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a continuous map. We call a function $F : \mathbb{R} \mapsto \mathbb{R}$ a lift of f if

$$\pi \circ F = f \circ \pi.$$

A lift F inherits regularity properties of f e.g. continuity, differentiability, smoothness. In the following we refer to maps of \mathbb{S}^1 with minuscule letters e.g. f . The corresponding capital letter F denotes a lift with the property $F(0) \in [0, 1)$, which is unique.

We recall that a continuous function $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ has degree one if for all $x \in \mathbb{R}$, $F(x+1) = F(x) + 1$. Moreover, f is said non-decreasing if its lift is non-decreasing.

1.3 Rotation number

Let $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a non-decreasing, continuous function of degree one then the limit

$$\lim_{n \rightarrow \infty} \frac{F^n(x)}{n}$$

exists for any $x \in \mathbb{R}$ and it is independent of x . This limit is called the *rotation number* of f and will be denoted by $\rho(f)$.

ρ as a function of f is continuous:

Proposition 1.3. *If $(f_n)_{n \in \mathbb{N}}$ is a family of continuous, non-decreasing functions of degree one and if $f_n \rightarrow f_0$ uniformly then $\rho(f_n) \rightarrow \rho(f_0)$.*

The rotation number is also non-decreasing:

Proposition 1.4. *If $F_1 < F_2$ then $\rho(f_1) \leq \rho(f_2)$. Moreover if $\rho(f_1)$ or $\rho(f_2)$ is irrational then $\rho(f_1) < \rho(f_2)$.*

2 Proof of Theorem 1.1

2.1 Technical Lemmas

Lemma 2.1. *Let $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a continuous, non-decreasing, degree one function with irrational rotation number $\rho \in [0, 1)$. Then the following statements are equivalent:*

1. f is a Denjoy counterexample,
2. f has not dense orbits,
3. f has a wandering interval, i.e. there exists a non-empty interval $I \subset \mathbb{S}^1$ such that, for all $n, m \in \mathbb{Z}$, $n \neq m$, $f^n(I) \cap f^m(I) = \emptyset$,
4. there exists an interval $I \subset \mathbb{S}^1$ such that $|I| > 0$ and $|f^n(I)| \rightarrow 0$ for $n \rightarrow +\infty$.

The proof of this lemma can be found in [4], p. 263.

In the following, for all $m \in \mathbb{N}$ the derivative of order m of a real function F in a point x will be denoted by $F^{(m)}(x)$.

For $F : \mathbb{R} \mapsto \mathbb{R}$, a j -times differentiable map we define

$$\|F\|_{C^j} = \sup_{x \in \mathbb{R}, 0 \leq i \leq j} |F^{(i)}(x)|$$

Lemma 2.2. *Let $n \in \mathbb{N}$ odd (resp. even) and let $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a C^n , non-decreasing, degree one function, for which there exists an interval $U = \pi(a, b)$ such that for all $x \in (a, b)$ and for all $m \in \mathbb{N}$, $F^{(m)}(x) = 0$.*

Then, $\forall \epsilon \in (0, \frac{1}{4})$, $\forall \delta \in (0, 1)$, there exists a C^n , non-decreasing, degree one function, $\tilde{f} = \tilde{f}_{n, \epsilon, \delta} : \mathbb{S}^1 \mapsto \mathbb{S}^1$, satisfying the following conditions:

1. $\|\tilde{F} - F\|_{C^n} < \delta$,
2. $\rho(\tilde{f}) = \rho(f)$,
3. $|\tilde{F}^{(1)}(x) - F^{(1)}(x)| < \delta F^{(1)}(x)$, $\forall x \notin (a - \frac{1}{4}(b - a), b + \frac{1}{4}(b - a))$,
4. for all $m \in \mathbb{N}$, $\tilde{F}^{(m)}(x) = 0 \Leftrightarrow x \in (a, a + \frac{\epsilon}{2^n}) \cup (a + \frac{2\epsilon}{2^n}, b)$ (resp. $x \in (a, b - \frac{2\epsilon}{2^n}) \cup (b - \frac{\epsilon}{2^n}, b)$).

Proof. The proof of this lemma is the same of the proof of Lemma 5 in [4] or Lemma 2.2 in [9]. \square

Lemma 2.3. *Let $n \in \mathbb{N}$ odd (resp. even) and let $\tilde{f} : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a C^n , non-decreasing, degree one function.*

We assume that there exist $\epsilon > 0$ and two intervals $I = (a, a + \frac{\epsilon}{2^n})$ (resp. $I = (a, b - \frac{2\epsilon}{2^n})$) and $J = (a + \frac{2\epsilon}{2^n}, b)$ (resp. $J = (b - \frac{\epsilon}{2^n}, b)$), such that for all $m \in \mathbb{N}$:

$$x \in I \cup J \Leftrightarrow \tilde{F}^{(m)}(x) = 0$$

and suppose that there exists a positive integer r such that $\tilde{f}^r(I) \in (b - \frac{\epsilon}{2^{n+1}}, b)$ (resp. $\tilde{f}^r(J) \in (a, a + \frac{\epsilon}{2^{n+1}})$).

Then, $\forall \sigma \in (0, 1)$, $\exists g = g_{n, \epsilon, \sigma} : \mathbb{S}^1 \mapsto \mathbb{S}^1$ a C^{n+1} , non-decreasing, degree one function satisfying the following conditions:

1. $\|G - \tilde{F}\|_{C^n} < \sigma$,
2. $\rho(g) = \rho(\tilde{f})$,
3. $|G^{(1)}(x) - \tilde{F}^{(1)}(x)| < \sigma \tilde{F}^{(1)}(x)$, $\forall x \notin (a - \frac{1}{4}(b - a), b + \frac{1}{4}(b - a))$,
4. $G^{(m)}(x) = 0 \Leftrightarrow x \in J$ (resp. I),
5. on some left-sided (resp. right-sided) neighborhood of $a + \frac{2\epsilon}{2^n}$ (resp. $b - \frac{2\epsilon}{2^n}$), f can be represented as $h_{l,n}((a + \frac{2\epsilon}{2^n} - x)^{n+2})$ (resp. $h_{r,n}((x - b - \frac{2\epsilon}{2^n})^{n+3})$) where $h_{l,n}$ (resp. $h_{r,n}$) is a C^∞ -diffeomorphism on an open neighborhood of $a + \frac{2\epsilon}{2^n}$ (resp. $b - \frac{2\epsilon}{2^n}$),
6. $G^r(I) \in (b - \frac{\epsilon}{2^{n+1}}, b)$ (resp. $G^r(J) \in (a, a + \frac{\epsilon}{2^{n+1}})$)

Proof. The proof of this lemma is the same of the proof of Lemma 6 in [4] or Lemma 2.3 in [9]. \square

2.2 Proof of Theorem 1.1

Proof. Let $\rho \in [0, 1)$ and $\epsilon \in (0, \frac{1}{4})$ be two fixed irrational numbers. We start with a \mathcal{C}^1 , non-decreasing, degree one, circle map f and an interval $U_0 = \pi(a_0, b_0)$ of length l such that

$$l > 2 \sum_{i=0}^{\infty} \frac{\epsilon}{2^i} \quad (2.4)$$

and

- $\forall m \in \mathbb{N}, F^{(m)}(x) = 0 \Leftrightarrow x \in (a_0, b_0)$,
- on some right-sided neighborhood of b_0 , f can be represented as $h_{r,0}((x - b_0)^2)$ where $h_{r,0}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of b_0 ,
- on some left-sided neighborhood of a_0 , f can be represented as $h_{l,0}((a_0 - x)^2)$ where $h_{l,0}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of a_0 .

For all $t \in [0, 1)$ we consider $F_t = F + t$ and we choose t_0 such that $\rho(f_{t_0}) = \rho$. The existence of such a t_0 is guaranteed by Proposition 1.3.

We denote by $f_0 = f_{t_0}$ and by I a proper subset of $f_0^{-1}(b_0 - \epsilon, b_0)$ having strictly positive length.

The Denjoy counterexample will be constructed as limit of a sequence of functions:

$$(f_n : \mathbb{S}^1 \mapsto \mathbb{S}^1)_{n \in \mathbb{N}},$$

for which there exist a sequence of arcs $\{U_n\}_{n \in \mathbb{N}}$ with

$$U_n = \pi((a_n, b_n)) \text{ and } U_n \subset U_{n-1}$$

and a sequence of integers $\{r_n\}_{n \in \mathbb{N}}$ fulfilling

$$1 = r_0 \text{ and } r_n < r_{n+1}$$

such that, for all $i \in \mathbb{N}$ the following conditions are satisfied:

1. f_i is a \mathcal{C}^i , non-decreasing, degree one map,
2. $\rho(f_i) = \rho$,
3. $\forall m \in \mathbb{N}, F_i^{(m)}(x) = 0$ if and only if $x \in (a_i, b_i)$,
4. if i is odd (resp. even) on some right-sided neighborhood of b_i , f can be represented as $h_{r,i}((x - b_i)^{i+3})$ (resp. $h_{r,i}((x - b_i)^{i+2})$) where $h_{r,i}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of b_i ,
5. if i is odd (resp. even) on some left-sided neighborhood of a_i , f can be represented as $h_{l,i}((a_i - x)^{i+1})$ (resp. $h_{l,i}((a_i - x)^{i+2})$) where $h_{l,i}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of a_i ,
6. $|U_i| = |U_{i-1}| - \frac{2\epsilon}{2^{i-1}}$,
7. $\|F_i - F_{i-1}\|_{\mathcal{C}^i} \leq \frac{1}{2^i}$,

8. $|F_i^{(1)}(x) - F_{i-1}^{(1)}(x)| < \frac{1}{2^{i+1}} F_i^{(1)}(x)$, $\forall x \notin (a_i - \frac{1}{4}(b_i - a_i), b_i + \frac{1}{4}(b_i - a_i))$,
9. $0 < |f_i^j(I)| < \frac{1}{2^{k-1}}$, if $r_{k-1} \leq j < r_k$ for $k \in \{1, 2, \dots, i\}$,
10. $f_i^j(I) \cap U_i = \emptyset$, if $0 \leq j < r_i$ and

$$f_i^{r_i}(I) \in \begin{cases} \pi(b_i - \frac{\epsilon}{2^i}, b_i) & \text{if } i \text{ is even} \\ \pi(a_i, a_i + \frac{\epsilon}{2^i}) & \text{if } i \text{ is odd} \end{cases}$$

The sequence $\{f_n\}_{n \in \mathbb{N}}$ will be constructed iteratively. Suppose that we have f_n . We produce f_{n+1} by perturbing f_n in the way that the conditions (1)-(10) are still satisfied. We suppose n odd (the case of n even is analogous), then $f_n^{r_n}(I) \in \pi(a_n, a_n + \frac{\epsilon}{2^n})$.

We fix $\delta \in (0, 1)$. By Lemma 2.2 applied to f_n we produce a \mathcal{C}^n , non-decreasing, degree one function $\tilde{f}_{n,\delta} := \tilde{f}_{n,\epsilon, \frac{\delta}{2^{n+2}}}$ such that $\forall m \in \mathbb{N}$,

$$\tilde{F}_{n,\delta}^{(m)}(x) = 0 \Leftrightarrow x \in (a_n, a_n + \frac{\epsilon}{2^n}) \cup (a_n + \frac{2\epsilon}{2^n}, b_n).$$

We study now the orbit of I under $\tilde{f}_{n,\delta}$. Observe that by construction $\tilde{F}_{n,\delta}^i \rightarrow F_n^i$ for $\delta \rightarrow 0$ uniformly for $i \in \{1, 2, \dots, r_n\}$, then we can fix $\delta' < \delta < 1$, such that:

$$|\tilde{f}_{n,\delta'}^j(I)| < \frac{1}{2^{k-1}} \text{ for all } r_{k-1} \leq j < r_k, \quad k \in \{1, 2, \dots, n\},$$

$$\tilde{f}_{n,\delta'}^j(I) \cap U_n = \emptyset \text{ for all } 0 \leq j < r_n \quad (2.5)$$

and

$$\tilde{f}_{n,\delta'}^{r_n}(I) \subset \pi(a_n, a_n + \frac{\epsilon}{2^n}). \quad (2.6)$$

Let us now study the orbits of $J_n = \pi(a_n, a_n + \frac{\epsilon}{2^n})$ under $\tilde{f}_{n,\delta'}$. We may have two different cases:

- there exists $m > 0$ such that $\tilde{f}_{n,\delta'}^m(J_n) \in U_n$,
- for all $m > 0$, $\tilde{f}_{n,\delta'}^m(J_n) \notin U_n$.

Observe that the second situation never occurs. In fact, in such a case we can construct $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ being non-decreasing function of degree one, which is equal to $\tilde{f}_{n,\delta'}$ everywhere except on $U_n \setminus \pi(a_n + \frac{\epsilon}{2^n}, b_n)$. Moreover, g can be chosen so that on some right-sided neighborhood of $a_n + \frac{\epsilon}{2^n}$ it is equal to $h_{r,n}((x - (a_n + \frac{\epsilon}{2^n}))^{n+1})$ for some \mathcal{C}^∞ -diffeomorphism $h_{r,n}$. Such g belongs to the class of functions with a flat interval (being J_n in this case) studied in [3] (see property (5)). We notice that $g^m(J_n) = \tilde{f}_{n,\delta'}^m(J_n)$ thus the orbits of g are not dense. By Lemma 2.1 it has a wandering interval and therefore contradicts Corollary to Theorem 1 in [3].

We study now the first case. The aim is to prove that there exists δ'' (smaller than δ' if necessary) such that

$$\tilde{f}_{n,\delta''}^m(J_n) \in \pi(b_n - \frac{\epsilon}{2^{n+1}}, b_n). \quad (2.7)$$

Because of the fact that the rotation number is irrational $f_n^m(U_n)$ does not enter into U_n and we may suppose that $f_n^m(U_n)$ approaches U_n from the right side. Observe that the functions $\tilde{f}_{n,\delta}$ approximate f_n , in particular

$$\tilde{f}_{n,\delta}^m(J_n) \rightarrow f_n^m(U_n),$$

as $\delta \searrow 0$. Moreover $\delta \mapsto \tilde{f}_{n,\delta}^m(J_n)$ is continuous and the set

$$A = \{\tilde{f}_{n,\delta}^m(J_n) : \delta \in [0, \delta']\},$$

is an interval of \mathbb{S}^1 containing $\tilde{f}_{n,\delta'}^m(J_n)$ and $f_n^m(U_n)$. The proof is concluded once we observe that, because of the fact that the rotation number ρ is irrational, $A \cap J_n = \emptyset$, so A covers a portion of the interior of $\pi(a_n + \frac{\epsilon}{2^n}, b_n)$. Because we are supposing that $f_n^m(U_n)$ approaches U_n from the right side, we can choose $\delta'' < \delta'$ if necessary such that $\tilde{f}_{n,\delta}^m(J_n)$ is contained in a sub-interval of $\pi(b_n - \frac{\epsilon}{2^{n+1}}, b_n)$.

We denote by $a_{n+1} = a_n + \frac{2\epsilon}{2^n}$, $b_{n+1} = b_n$ and consequently $U_{n+1} = \pi((a_n + \frac{2\epsilon}{2^n}, b_n))$. We fix $\sigma > 0$ and we apply Lemma 2.3 to the function $\tilde{f}_{n,\delta''}$. We get a \mathcal{C}^{n+1} , non-decreasing, degree one map $f_{n+1,\sigma} = f_{n+1,\epsilon,\frac{\sigma}{2^{n+1}}} : \mathbb{S}^1 \mapsto \mathbb{S}^1$ such that for all $m \in \mathbb{N}$,

$$F_{n+1,\sigma}^{(m)}(x) = 0 \Leftrightarrow x \in (a_{n+1}, b_{n+1}).$$

Finally we define $r_{n+1} = r_n + m$. Since $f_{n+1,\sigma}^i \rightarrow \tilde{f}_n^i$ uniformly for all $i \in \{1, 2, \dots, r_n + m\}$ and since $\tilde{f}_n^i(I)$ is a singleton for $i > r_n$, then we get that (we set $f_{n+1} = f_{n+1,\sigma}$):

$$|f_{n+1}^j(I)| < \frac{1}{2^{k-1}}$$

for all $r_{k-1} \leq j < r_k$, with $k \in \{1, 2, \dots, n\}$, and

$$|f_{n+1}^j(I)| < \frac{1}{2^n}$$

if $r_n \leq j < r_{n+1}$.

By (2.5), (2.6) and (2.7), since $f_{n+1,\sigma}^i \rightarrow \tilde{f}_n^i$ uniformly for all $i \in \{1, 2, \dots, r_{n+1}\}$, then

$$f_{n+1}^i(I) \cap U_{n+1} = \emptyset \text{ for all } 0 \leq i < r_{n+1}$$

and

$$f_{n+1}^{r_{n+1}}(I) \subseteq \pi(b_n - \frac{\epsilon}{2^{n+1}}, b_n).$$

So we have constructed a sequence of functions $(f_k)_{k \in \mathbb{N}}$ satisfying (1)-(10) for all $n \geq 1$. By condition (7) the sequence $(f_k)_{k \in \mathbb{N}}$ converges in the sense of the norm $\|\cdot\|_{\mathcal{C}^n}$ for all n . Then the limit function f is a \mathcal{C}^∞ , non-decreasing, degree one circle map which has rotation number ρ (see (1), (2), (3), (4) and (5)). Let consider now $a = \lim_n a_n$ and $b = \lim_n b_n$ and $U = \pi(a, b)$. By (3) and (8) $F^{(1)}(x) = 0$ if and only if $x \in (a, b)$ and by (6) and the assumption on the size of U_0 (see 2.4), U has strictly positive length.

Finally, by conditions (9) and (10),

$$|f_n^i(I)| \rightarrow 0 \text{ for } i \rightarrow +\infty$$

uniformly in n , and then

$$|f^i(I)| \rightarrow 0 \text{ if } i \rightarrow +\infty.$$

By Lemma 2.1 f has a wandering interval and thus it is not conjugate to a rotation. \square

3 Applications: Cherry Flows

3.1 Basic Definitions

Let X be a C^∞ vector field on the torus T^2 . Denote the flow through a point x by $t \rightarrow X_t(x)$.

The ω -limit set of a positive semi-trajectory $\gamma^+(x)$ is the set

$$\omega(\gamma^+(x)) = \{y : \exists t_n \rightarrow \infty \text{ with } X_{t_n}(x) \rightarrow y\},$$

and α -limit set of a negative semi-trajectory $\gamma^-(x)$ is

$$\alpha(\gamma^-(x)) = \{y : \exists t_n \rightarrow \infty \text{ with } X_{-t_n}(x) \rightarrow y\}.$$

The ω -limit set (α -limit set) of any positive (negative) semi-trajectory of the trajectory γ is called ω -limit set $\omega(\gamma)$ of γ (α -limit set $\alpha(\gamma)$ of γ).

The trajectory is ω -recurrent (α -recurrent), if it is contained in its ω -limit set (α -limit set). The trajectory is *recurrent* if it is both ω -recurrent and α -recurrent. A recurrent trajectory is *non-trivial* if it is neither a fixed point nor a periodic trajectory.

Definition 3.1. A *Cherry flow* is a C^∞ flow on the torus \mathbb{T}^2 without closed trajectories which has exactly two singularities, a sink and a saddle, both hyperbolic.

The first example of such a flow was given by Cherry in [1].

3.2 Basic Properties of Cherry Flows

We state now basic properties of Cherry flows. For more details the reader can refer to [8], [6], [11].

Given a Cherry flow X we can always find a circle \mathbb{S}^1 which is not retractable to a point and is everywhere transverse to X . It is called a Poincaré section.

The first return map f of X to \mathbb{S}^1 turns out to be a continuous degree one circle map which is constant on an interval U (it corresponds to the interval of points which are attracted to the singularities).

Definition 3.2. Let γ be a non-trivial recurrent trajectory. Then the closure $\overline{\gamma}$ of γ is called a *quasi-minimal set*.

Every Cherry flow has only one quasi-minimal set which is locally homeomorphic to the Cartesian product of a Cantor set Ω and a segment I . Moreover Ω is equal to the non-wandering set² of the first return function f .

We give now the definition of attractor. For more details on the concept of attractor the reader can refer to [7].

Definition 3.3. Let M be a smooth compact manifold. A closed subset $A \subset M$ is called an *attractor* if it satisfies two conditions:

1. the realm of attraction $\Lambda(A)$ consisting of all points $x \in M$ for which $\omega(x) \subset A$, has strictly positive Lebesgue measure;

²the set of the points x such that for any open neighborhood $V \ni x$ there exists an integer $n > 0$ such that the intersection of V and $f^n(V)$ is non-empty.

2. *there is no strictly smaller closed set $A' \subset A$ so that $\Lambda(A')$ coincides with $\Lambda(A)$ up to a set of zero Lebesgue measure.*

The first condition says that there is some positive possibility that a randomly chosen point will be attracted to A , and the second says that every part of A plays an essential role.

We are now ready to construct an example of Cherry flow whose quasi-minimal set is a attractor.

3.3 Proof of Theorem 1.2

Proof. Let ρ be an irrational number and let f be the Denjoy counterexample constructed in Theorem 1.1 having rotation number ρ . By suspending f , we get a Cherry flow X having first return map f . We claim that the quasi-minimal set Q of X is an attractor. Let \mathbb{S}^1 be the Poincaré section and let $I \subset \mathbb{S}^1$ be the wandering interval of f . Then $\omega(I) \subset Q$ and the realm of attraction $\Lambda(Q)$ has positive Lebesgue measure. Point 1 of Definition 3.3 is satisfied.

For point 2, by Definition 3.2, $Q = \overline{\gamma}$ with γ a non-trivial recurrent orbit. In particular $\gamma \subset \omega(\gamma) \subset \overline{\gamma}$. Let $Q' \subset Q$ be a closed set such that $\Lambda(Q') = \Lambda(Q)$. Then $\gamma \subset Q'$ and consequently $Q = Q'$. The proof is now complete. \square

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4 Appendix

The aim of this section is to show that the techniques used for proving Theorem 1.1 composed with the ones of [9] are robust and allow to produce more Denjoy counterexamples. We can in fact prove that:

Theorem 4.1. *Let p be a point on the circle. For all irrational numbers $\rho \in [0, 1)$ there exists a circle homeomorphism $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with rotation number ρ and an arc of the form $U = (p, \cdot)$ (resp. $U = (\cdot, p)$) which satisfy the following properties:*

- *f is piecewise \mathcal{C}^∞ ,*
- *f is a \mathcal{C}^∞ map on $\mathbb{S}^1 \setminus \{p\}$*
- *f is constant on U ,*
- *p is a flat half-critical point³ for f ,*
- *f is a Denjoy counterexample.*

³For a piecewise \mathcal{C}^∞ function $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, a point $p \in \mathbb{S}^1$ is said *flat half-critical* if the left (resp. right) derivative of f at p is not zero and the right (resp. left) derivatives of any order are zero.

Proof. We give just an outline of the proof.

We fix $p \in \mathbb{S}^1$ and we denote by $p' = \pi^{-1}(p)$. Let $\rho \in [0, 1)$ and let $\epsilon \in (0, \frac{1}{4})$ be two fixed irrational numbers.

We consider a piecewise \mathcal{C}^∞ , non-decreasing, degree one, circle map f_0 and an interval $U_0 = \pi(p', b_0)$ of the length l such that

$$l > 2 \sum_{i=0}^{\infty} \frac{\epsilon}{2^i}$$

such that

- $\forall m \in \mathbb{N} \ F_0^{(m)}(x) = 0 \Leftrightarrow x \in (p', b_0)$,
- the left derivative of f_0 , $F_{0-}^{(1)}(p') > K > 0$,
- on some right-sided neighborhood of b_0 , f_0 can be represented as $h_{r,0}((x - b_0)^2)$ where $h_{r,0}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of b_0 .

We denote by J a proper subset of $f_0^{-1}(b_0 - \epsilon, b_0)$ having strictly positive length.

The Denjoy counterexample will be defined as the limit of a sequence of functions:

$$(f_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1)_{n \in \mathbb{N}},$$

for which there exist a sequence of arcs $\{U_n\}_{n \in \mathbb{N}}$ with

$$U_n = \pi((p', b_n)) \text{ and } U_n \subset U_{n-1}$$

and a sequence of integers $\{r_n\}_{n \in \mathbb{N}}$

$$1 = r_0 \text{ and } r_n < r_{n+1}$$

such that, for all $i \in \mathbb{N}$ the following conditions are satisfied:

1. f_i is a piecewise \mathcal{C}^∞ , non-decreasing, degree one map,
2. $\rho(f_i) = \rho$,
3. $\forall m \in \mathbb{N}, F_i^{(m)}(x) = 0$ if and only if $x \in (p', b_i)$,
4. the left derivative of f_i , $F_{i-}^{(1)}(p') > K > 0$,
5. on some right-sided neighborhood of b_i , f can be represented as $h_{r,i}((x - b_i)^{i+2})$ where $h_{r,i}$ is a \mathcal{C}^∞ -diffeomorphism on an open neighborhood of b_i ,
6. $|U_i| = |U_{i-1}| - \frac{2\epsilon}{2^{i-1}}$,
7. $\|F_i - F_{i-1}\|_{\mathcal{C}^i} < \frac{1}{2^i}$,
8. $|F_i^{(1)}(x) - F_{i-1}^{(1)}(x)| < F_i^{(1)}(x) \frac{1}{2^{i+1}}, \forall x \notin (p' - \frac{1}{4}(b_i - p'), b_i + \frac{1}{4}(b_i - p'))$,
9. $0 < |f_i^j(J)| < \frac{1}{2^{k-1}}$, if $r_{k-1} \leq j < r_k$ for $k \in \{1, 2, \dots, i\}$,
10. $f_i^j(J) \cap U_i = \emptyset$, if $0 \leq j < r_i$ and $f_i^{r_i}(J) \subseteq \pi(b_i - \frac{\epsilon}{2^i}, b_i)$.

The existence of such a sequence can be proved following the steps of the proofs of Theorem 1.1 and Theorem 1.6 in [9].

The limit function will then has the properties stated in the theorem. In particular it will be a Denjoy counterexample by points (9), (10) and Lemma 2.1. \square

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